Spatial Extent of Branching Brownian Motion

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Extreme Value Statistics

- Extreme value statistics has been growing in prominence.

- In many real world examples the extreme value is not independent of the rest of the set and there are **strong correlations between near-extreme values**.

- Examples include **extreme temperatures** as part of heat or cold waves, **earthquakes and financial crashes** where extreme fluctuations are accompanied by foreshocks and aftershocks.

- Particularly important in **disordered systems** where energy levels near the ground state become important at low but finite temperature.

- Although EVS of independent identically distributed (i.i.d.) variables are fully understood, **very few analytical results for strongly correlated random variables**.
At each time step $[t, t + \Delta t]$ the particle can:

- **A)** die with probability $a\Delta t$

- **B)** split into two independent particles with probability $b\Delta t$

- **C)** diffuse by a distance $\Delta x = \eta(t)\Delta t$, with probability $1 - (a + b)\Delta t$.

\[
\langle \eta(t) \rangle = 0, \quad \langle \eta(t_1)\eta(t_2) \rangle = 2D\delta(t_1 - t_2) \tag{1}
\]
Figure: A realization of the dynamics of branching Brownian motion with death (left) in the supercritical regime ($b > a$) and (right) in the critical regime ($b = a$). The particles are numbered sequentially from right to left as shown in the inset.
The Backward Fokker-Planck Approach

- We look at the contribution from the **first time step** $[0, \Delta t]$ to the final time step $t + \Delta t$
Number of Particles in the system

- $P(n, t) =$ Probability there are exactly $n$ particles at time $t$.

- Using the Backward Fokker-Planck approach

  $$P(n, t + \Delta t) = [1 - (a + b)\Delta t]P(n, t) + b\Delta t \sum_{m=0}^{n} P(m, t)P(n - m, t) + a\Delta t \delta_{n,0}. \quad (2)$$

- In the $\Delta t \to 0$ we have

  $$\frac{\partial P(n, t)}{\partial t} = -(a + b)P(n, t) + b \sum_{m=0}^{n} P(m, t)P(n - m, t) + a \delta_{n,0}. \quad (3)$$

- We can solve this using standard generating functions.
Number of Particles in the system

- The solutions are

\[ P(0, t) = \frac{a(e^{bt} - e^{at})}{be^{bt} - ae^{at}}, \quad P(n \geq 1, t) = (b - a)^2 e^{(a+b)t} \frac{b^{n-1}(e^{bt} - e^{at})^{n-1}}{(be^{bt} - ae^{at})^{n+1}}. \]  

(4)

- In the critical regime \((b = d)\) this reduces to

\[ P(0, t) = \frac{bt}{1 + bt}, \quad P(n \geq 1, t) = \frac{(bt)^{n-1}}{(1 + bt)^{n+1}}. \]  

(5)

- The average number of particles is

\[ \langle N(t) \rangle = e^{(b-a)t}. \]  

(6)
The Maximum of BBM

- $Q(X, t) =$ Probability that $X_{\text{max}} \leq X$ at time $t$

- PDF of $X_{\text{max}}$:
  
  \[ P_{\text{marg}}(X, t) = \frac{\partial}{\partial X} Q(X, t). \]  

- The initial condition is
  
  \[ Q(X, 0) = \theta(X) \]  

- The boundary conditions are
  
  \[ Q(X, t) = \begin{cases} 
  1 & \text{for} \quad X \to \infty \\
  0 & \text{for} \quad X < 0. 
  \end{cases} \]  

Using the backward Fokker-Planck approach, we have

\[
Q(X, t + \Delta t) = (1 - (a + b)\Delta t) \langle Q(X - \eta(0)\Delta t, t) \rangle_{\eta(0)} + b\Delta t Q^2(X, t) + a\Delta t.
\] (10)

In the $\Delta t \rightarrow 0$ we have

\[
\frac{\partial Q(X, t)}{\partial t} = D \frac{\partial^2 Q(X, t)}{\partial X^2} - (a + b)Q(X, t) + bQ^2(X, t) + a
\] (11)

In terms of $R(X, t) = 1 - Q(X, t)$:

\[
\frac{\partial R(X, t)}{\partial t} = D \frac{\partial^2 R(X, t)}{\partial X^2} + (b - a) R(X, t) - b R^2(X, t),
\] (12)

Non-linear equation with no known general solution ($a = 0$ is the famous Fisher-Kolmogorov-Pertrovski-Piscounov Equation).
Dimensionless Variables

It is natural to consider the evolution equations in terms of dimensionless variables as follows

\[ x = \frac{X}{\sqrt{D/b}}, \quad \left( y = \frac{Y}{\sqrt{D/b}}, \quad s = \frac{\zeta}{\sqrt{D/b}} \right), \]

\[ \tau = bt, \]

\[ \Delta = \frac{a}{b} - 1. \] (13)

In terms of these dimensionless variables Eq. (12) takes the simpler form

\[ \frac{\partial R(x, \tau)}{\partial \tau} = \frac{\partial^2 R(x, t)}{\partial x^2} - \Delta R(x, y, \tau) - R^2(x, y, \tau). \] (14)

For \( \Delta > 0 \): expected to approach a stationary limit as \( \tau \to \infty \):

\[ R(x) = R(x, y, \tau \to \infty). \] (15)
In the critical case $\Delta = 0$ (S. Sawyer and J. Fleischman, Proc. Natl. Acad. Sci. USA 76(2), 87 (1979)):

$$R(x) = \frac{1}{\left(1 + \frac{x}{\sqrt{6}}\right)^2}.$$  \hspace{1cm} (16)

Consequently, $p_{\text{marg}}(x) = -dR(x)/dx$ has the asymptotic behaviors

$$p_{\text{marg}}(x) \sim \begin{cases} 
  p_{\text{marg}}(0) = \sqrt{\frac{2}{3}}, & x \to 0 \\
  \frac{12}{x^3}, & x \to \infty.
\end{cases}$$ \hspace{1cm} (17)
In the subcritical case $\Delta > 0$ (S. Sawyer and J. Fleischman, Proc. Natl. Acad. Sci. USA 76(2), 87 (1979)):

$$R(x) = \frac{3\Delta}{2} \text{csch}^2 \left( \frac{\sqrt{\Delta}}{2} x + \sinh^{-1} \sqrt{\frac{3\Delta}{2}} \right).$$  \hspace{1cm} (18)

Consequently, $p_{\text{marg}}(x) = -dR(x)/dx$ has the asymptotic behaviors

$$p_{\text{marg}}(x) \sim \begin{cases} 
 p_{\text{marg}}(0) = \sqrt{\frac{2}{3}} + \Delta, & x \to 0 \\
 6\Delta^{\frac{3}{2}} e^{-2 \sinh^{-1} \sqrt{\frac{3\Delta}{2}}} \exp(-\sqrt{\Delta} x), & x \to \infty.
 \end{cases}$$ \hspace{1cm} (19)
Figure: Schematic representation of a trajectory of the BBM confined in the box $[-Y, X]$. Note that $X_{\text{max}}$ and $X_{\text{min}}$ denote respectively the maximum and the minimum of the process up to time $t$. The process starts with a single particle at the origin at time $t = 0$ and hence $X_{\text{max}} \geq 0$ while $X_{\text{min}} \leq 0$. 
Joint Distribution of the Maximum and Minimum

- \( Q(X, Y, t) = \) Joint probability that \( X_{\text{max}} \leq X \) AND \( X_{\text{min}} \geq -Y \) at time \( t \)

- Using the backward Fokker-Planck approach, we have

\[
Q(X, Y, t + \Delta t) = b\Delta t \ Q^2(X, Y, t) + a\Delta t \\
+ (1 - (b + a)\Delta t) \langle Q(X - \Delta x, Y + \Delta x, t) \rangle_{\eta(0)},
\]

(20)

- In the limit \( \Delta t \to 0 \), we arrive at the exact BFP evolution equation

\[
\frac{\partial}{\partial t} Q(X, Y, t) = D \left( \frac{\partial}{\partial X} - \frac{\partial}{\partial Y} \right)^2 Q(X, Y, t) + a \\
-(b + a) \ Q(X, Y, t) + b \ Q^2(X, Y, t).
\]

(21)
The initial condition is

\[ Q(X, Y, 0) = \Theta(X)\Theta(Y) , \]

(22)

The boundary conditions are

\[ Q(X, Y, t) = \begin{cases} 
0 & \text{for} \quad X < 0, \\
0 & \text{for} \quad Y < 0 . 
\end{cases} \]

(23)
In terms of the complimentary probability \( R(X, Y, t) = 1 - Q(X, Y, t) \), we have

\[
\frac{\partial R(X, Y, t)}{\partial t} = D \left( \frac{\partial}{\partial X} - \frac{\partial}{\partial Y} \right)^2 R(X, Y, t) + (b - a) R(X, Y, t) - b R^2(X, Y, t),
\]

(24)

with the initial conditions

\[
R(X, Y, 0) = \Theta(-X)\Theta(-Y).
\]

(25)

and the boundary conditions

\[
R(X, Y, t) = \begin{cases} 
1 & \text{for } X < 0, \\
1 & \text{for } Y < 0.
\end{cases}
\]

(26)
Figure: The solutions $R(s, v, t)$ at different times obtained by numerical integration, with $dx = 0.1$, $dt = 0.001$, $D = 1$ and $b = 0.5$. We find that at large times this converges to a stationary bivariate function $R(s, v)$. 
The distribution of the span $s = X_{\text{max}} - X_{\text{min}}$ is given by

$$P(s, t) = \int_0^\infty \int_0^\infty dXdY \delta(X + Y - s)P(X, Y, t). \quad (27)$$

In terms of the dimensionless variables

$$P(\zeta, \tau) = \int_0^\infty \int_0^\infty dx dy \delta(x + y - \zeta)P(x, y, \tau). \quad (28)$$

\[ p_{\text{uncorr}}(\zeta) = \int_0^\zeta p_{\text{marg}}(x)p_{\text{marg}}(\zeta - x)dx . \quad (29) \]
Next, it is convenient to make a change of variables

$$\zeta = x + y,$$

$$v = x - y,$$

(30)

with $\zeta \in [0, \infty)$ and $v \in [-\zeta, \zeta]$. $\zeta$ represents the dimensionless span of the process.
Figure: Graph depicting the stationary joint cumulative probability $\mathcal{R}(x, y)$. The limiting distributions correspond to the marginal probabilities of the maximum and minimum $\mathcal{R}(x) = \mathcal{R}(x, y \to \infty)$ and $\mathcal{R}(y) = \mathcal{R}(x \to \infty, y)$ respectively.
In terms of these new variables Eq. (24) becomes

$$4 \left( \frac{\partial}{\partial v} \right)^2 R(\zeta, v) - \Delta R(\zeta, v) - R^2(\zeta, v) = 0,$$

valid in the regime $v \in [-\zeta, +\zeta]$ and $\zeta \in [0, +\infty)$. 

For a fixed $\zeta$, Monotonically decreasing $v \in [-\zeta, 0]$, Monotonically increasing $v \in [0, \zeta]$. 

Assuming analyticity around the minimum at $v = 0$ gives the condition

$$\left. \frac{\partial R(\zeta, v)}{\partial v} \right|_{v=0} = 0.$$
**Figure:** $R(\zeta, v)$ as a function of $v \in [-\zeta, +\zeta]$ for different values of $\zeta$. For fixed $\zeta$, $R(\zeta, v)$ is a smooth non-monotonic function, symmetric around $v = 0$ in $-\zeta \leq v \leq +\zeta$, and has a minimum at $v = 0$. 
Fortunately, Eq. (31) can be integrated with respect to \( v \) upon multiplying by a factor \( 2 \frac{\partial \mathcal{R}(\zeta, v)}{\partial v} \), yielding

\[
\left( \frac{\partial \mathcal{R}(\zeta, v)}{\partial v} \right)^2 = \frac{\Delta}{4} \mathcal{R}^2(\zeta, v) + \frac{1}{6} \mathcal{R}^3(\zeta, v) + \kappa(\zeta),
\]

(33)

where \( \kappa(\zeta) \) is a yet unknown integration constant.

To fix \( \kappa(\zeta) \), we use the condition in Eq. (32) and arrive at

\[
\left( \frac{\partial \mathcal{R}(\zeta, v)}{\partial v} \right)^2 = \frac{\Delta}{4} \left( \mathcal{R}^2(\zeta, v) - \mathcal{R}^2(\zeta, 0) \right) + \frac{1}{6} \left( \mathcal{R}^3(\zeta, v) - \mathcal{R}^3(\zeta, 0) \right)
\]

(34)
This equation can be conveniently expressed as

$$\frac{1}{\sqrt{\mathcal{R}(\zeta, 0)}} \mathcal{G} \left( \frac{3\Delta/2}{\mathcal{R}(\zeta, 0)}, \frac{\mathcal{R}(\zeta, \nu)}{\mathcal{R}(\zeta, 0)} \right) = \frac{\nu}{\sqrt{6}},$$

where the bivariate function $\mathcal{G}$ is defined by the integral

$$\mathcal{G}(\gamma, z) = \int_1^z \frac{dx}{\sqrt{(x^3 - 1) + \gamma (x^2 - 1)}},$$

The above function $\mathcal{G}(\gamma, z)$ can then be expressed as

$$\mathcal{G}(\gamma, z) = \frac{1}{(3 + 2\gamma)^{1/4}} \mathbf{F} \left[ \tan^{-1} \sqrt{\frac{z - 1}{\sqrt{3 + 2\gamma}}}, \frac{2\sqrt{3 + 2\gamma} - (3 + \gamma)}{4\sqrt{3 + 2\gamma}} \right],$$

where $z \geq 1$, $\gamma \geq 0$ and $\mathbf{F}$ is the elliptic integral of the first kind.

$$\mathbf{F}(\phi, k) = \int_0^\phi \frac{d\theta}{\sqrt{1 - k \sin^2 \theta}}$$
Next, inserting the boundary condition $\mathcal{R}(\zeta, \pm \zeta) = 1$ in the above equation we have

$$\frac{1}{\sqrt{\mathcal{R}(\zeta, 0)}} g \left( \frac{3\Delta/2}{\mathcal{R}(\zeta, 0)}, \frac{1}{\mathcal{R}(\zeta, 0)} \right) = \frac{\zeta}{\sqrt{6}}. \quad (39)$$

This is an **implicit equation** for $\mathcal{R}(\zeta, 0)$, the solution of which can then be injected in Eq. (35) to solve for $\mathcal{R}(\zeta, v)$ for all $\zeta$ and $v$. 


Figure: The function $G(\gamma, z)$ for different values of $\gamma$. For large $z$, $G(\gamma, z)$ saturates to a $\gamma$ dependent constant $C(\gamma)$. The case $\gamma = 0$ corresponds to the function $G(0, z)$ analyzed in the critical case. The limiting behaviors are $G(0, z) \to 0$ as $z \to 1$ and $G(0, z) \to C^* = \frac{\sqrt{\pi}}{3} \frac{\Gamma(\frac{1}{6})}{\Gamma(\frac{2}{3})} \approx 2.4286$ as $z \to \infty.$
Results

Figure: The function $\mathcal{R}(\zeta, 0)$ versus $\zeta$ in the critical regime, showing the limiting behaviors $\mathcal{R}(\zeta, 0) \to 1$ as $\zeta \to 0$ and $\mathcal{R}(\zeta, 0) \to \frac{B}{\zeta^2}$ as $\zeta \to \infty$ (dashed line). $B \approx 35.3901$. Inset: Plot of $1 - \mathcal{R}(\zeta, 0)$ showing the limiting behavior $1 - \mathcal{R}(\zeta, 0) \sim \left(\frac{1}{8}\right) \zeta^2$ as $\zeta \to 0$ (dashed line).
Figure: Asymptotic behavior of $\mathcal{R}(\zeta, 0)$ in the subcritical regime. The plot shows $\mathcal{R}(\zeta, 0)$ for different values of $\Delta$. The dashed lines representing the asymptotic exponential behavior $\mathcal{R}(\zeta, 0) \sim A \exp\left(-\sqrt{\Delta} \frac{\zeta}{2}\right)$ as $\zeta \to \infty$ are indistinguishable from the theoretically obtained curves as they match exactly. **Inset:** Plot of $1 - \mathcal{R}(\zeta, 0)$ showing the limiting behavior $1 - \mathcal{R}(\zeta, 0) \sim \frac{1}{8} (1 + \Delta) \zeta^2$ as $\zeta \to 0$ (dashed lines).
$\zeta \to 0$ Asymptotics

- To leading order in $\zeta$ we have
  \[ R(\zeta, 0) = 1 - \frac{1}{8} (1 + \Delta) \zeta^2 + O(\zeta^4). \]  
  (40)

- Therefore
  \[ R(\zeta, v) = 1 - \frac{1}{8} (1 + \Delta) (\zeta^2 - v^2) + O(\zeta^4, v^4). \]  
  (41)

- And hence
  \[ p(\zeta, v) = \frac{1}{2} \left( \frac{\partial^2}{\partial v^2} - \frac{\partial^2}{\partial \zeta^2} \right) R(\zeta, v) = \frac{1}{4} (1 + \Delta) + O(\zeta^2, v^2). \]  
  (42)

- And finally
  \[ p(\zeta) = \frac{1}{2} (1 + \Delta) \zeta + O(\zeta^3), \]  
  (43)

- To be compared with
  \[ p_{\text{uncorr}}(\zeta) \sim \left( \frac{2}{3} + \Delta \right) \zeta, \text{ when } \zeta \to 0. \]  
  (44)
Asymptotic behavior of $p(\zeta)$ for $\zeta \to \infty$

$\zeta \to \infty$ Asymptotics: Critical

- we obtain

$$R(\zeta, 0) = \frac{B}{\zeta^2} + O\left(\frac{1}{\zeta^4}\right), \quad (45)$$

where

$$B = 6 \, C^*^2 \approx 35.3901, \quad \text{with} \quad C^* = \frac{\sqrt{\pi} \, \Gamma\left(\frac{1}{6}\right)}{3 \, \Gamma\left(\frac{2}{3}\right)}. \quad (46)$$

- Hence, in the scaling limit $\zeta \to \infty, \nu \to \infty$ keeping $\zeta/\nu$ fixed

$$G\left(0, \frac{R(\zeta, \nu)}{R(\zeta, 0)}\right) = C^* \frac{\nu}{\zeta}. \quad (47)$$

- Inverting the above Eq. (47), we get

$$\frac{R(\zeta, \nu)}{R(\zeta, 0)} = F\left(C^* \frac{\nu}{\zeta}\right), \quad (48)$$

where $F(z)$ is defined as the inverse function of $G(0, z)$. 

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Results

Asymptotic behavior of \( p(\zeta) \) for \( \zeta \to \infty \)

\[
\mathcal{R}(\zeta, v) = \frac{B}{\zeta^2} \mathcal{F}\left(\frac{C^*v}{\zeta}\right). \tag{49}
\]

Inserting this expression into the expression for \( p(\zeta, v) \) yields

\[
p(\zeta, v) = -\frac{B}{2} \left[ \frac{6}{\zeta^4} \mathcal{F}\left(\frac{C^*v}{\zeta}\right) + \frac{6C^*v}{\zeta^5} \mathcal{F}'\left(\frac{C^*v}{\zeta}\right) \right.
\]
\[
+ \left. C^*^2 \left( \frac{v^2}{\zeta^6} - \frac{1}{\zeta^4} \right) \mathcal{F}''\left(\frac{C^*v}{\zeta}\right) \right]. \tag{50}
\]

The span distribution is then

\[
p(\zeta) = -\frac{1}{\zeta^3} \left( \frac{B}{C^*} \right) \int_0^{C^*} \text{d}z \left[ \frac{6}{\zeta^3} \mathcal{F}(z) + 6z\mathcal{F}'(z) + \left( z^2 - C^*^2 \right) \mathcal{F}''(z) \right]. \tag{51}
\]

Hence, we obtain

\[
p(\zeta) \sim \frac{A}{\zeta^3} \quad \text{for large } \zeta, \quad \tag{52}
\]
We can integrate this exactly!

\[ A = 8\pi \sqrt{3} = 43.53118 \ldots . \]  \hspace{1cm} (53)

Thus the leading asymptotic behavior for large \( \zeta \) is

\[ p(\zeta) \sim \frac{8\pi \sqrt{3}}{\zeta^3}. \]  \hspace{1cm} (54)

To be compared with

\[ \rho_{\text{uncorr}}(\zeta) \sim \frac{24}{\zeta^3}. \]  \hspace{1cm} (55)
Results

Asymptotic behavior of $p(\zeta)$ for $\zeta \to \infty$

\[
p(\zeta) \sim A \zeta^{-3}
\]

$p_{\text{uncorr}}(\zeta) \sim \left(\frac{1}{2}\right) \zeta$

\textbf{Figure}: Theoretical stationary PDF of the dimensionless span $p(\zeta)$ (solid line) in the critical regime.
\( \zeta \to \infty \) Asymptotics: Subcritical

- Here we get

\[
p(\zeta) \sim \frac{A^2}{2} \zeta \exp \left( -\sqrt{\Delta} \zeta \right), \quad \zeta \to \infty.
\]  

(56)

where

\[
A = 12 \Delta \exp \left[ -2 \sinh^{-1} \left( \sqrt{3\Delta/2} \right) \right].
\]

- To be compared with

\[
p_{\text{uncorr}}(\zeta) \sim \frac{\Delta}{4} A^2 \zeta \exp \left( -\sqrt{\Delta} \zeta \right).
\]  

(57)
Asymptotic behavior of $p(\zeta)$ for $\zeta \to \infty$.

Figure: Theoretical stationary PDF of the dimensionless span $p(\zeta)$ in the subcritical regime.
Monte Carlo Simulations

Figure: a. Probability distribution function of the dimensionless span $p(\zeta)$ extracted from Monte Carlo simulations (open circles) in the critical case ($\Delta = 0$). Here $t = 100$, $D = 1$, $a = b = 1$, and $dt = 0.0001$. The data is averaged over $5 \times 10^7$ realizations. b. Probability distribution function of the dimensionless span $p(\zeta)$ extracted from Monte Carlo simulations (open circles) in the subcritical regime. Here $t = 100$, $D = 1$, $a = 2$, $b = 1$ (i.e. $\Delta = 1$), and $dt = 0.0001$. 
Figure: Finite time span PDF $P(s, t)$ obtained from Monte Carlo simulations at different times with $dt = 0.0001$, $D = 1$ and $b = 0.5$. The data is averaged over $5 \times 10^7$ realizations. The bold lines represent the PDFs obtained from our numerical integration of the two dimensional non-linear partial differential equation. We find a perfect agreement between the PDFs obtained by both techniques.
Conclusion

- We obtained exact **analytical results for the span distribution** of one dimensional BBM.

- This was possible by **looking at the stationary regime**.

- We found that **correlations between the maximum and minimum persist in the stationary regime**.

- It will be interesting to extend our analysis to **convex hulls in higher dimensional BBM**.
Thank You.