Correlated Extreme Values in Branching Brownian Motion

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Branching processes are prototypical models of systems where **new particles are generated at every time step.**

Well studied in the context of **evolution, epidemic spreads, nuclear reactions** amongst others.

Related to several models such as continuum limit of **branching-annihilating-random-walk** (DP Universality), **GREM.**

Used in the modelling of **disordered systems and spin-glasses** where energy levels are random variables.
Branching Brownian Motion

At each time step $[t, t + \Delta t]$ the particle can:

- **A)** die with probability $d\Delta t$
- **B)** split into two independent particles with probability $b\Delta t$
- **C)** diffuse by a distance $\Delta x = \eta(t)\Delta t$, with probability $1 - (b + d)\Delta t$.

\[
\langle \eta(t) \rangle = 0, \quad \langle \eta(t_1)\eta(t_2) \rangle = 2D\delta(t_1 - t_2) \quad (1)
\]
Figure: A realization of the dynamics of branching Brownian motion with death (left) in the supercritical regime \((b > d)\) and (right) in the critical regime \((b = d)\). The particles are numbered sequentially from right to left as shown in the inset.
Extreme Value Statistics

- Extreme value statistics has been growing in prominence.

- In many real world examples the extreme value is not independent of the rest of the set and there are strong correlations between near-extreme values.

- Examples include extreme temperatures as part of heat or cold waves, earthquakes and financial crashes where extreme fluctuations are accompanied by foreshocks and aftershocks.

- Particularly important in disordered systems where energy levels near the ground state become important at low but finite temperature.

- Although EVS of independent identically distributed (i.i.d.) variables are fully understood, very few analytical results for strongly correlated random variables.
We look at the contribution from the first time step \([0, \Delta t]\) to the final time step \(t + \Delta t\).
Number of Particles in the system

- \( P(n, t) = \) Probability there are exactly \( n \) particles at time \( t \).

- Using the Backward Fokker-Planck approach

\[
P(n, t + \Delta t) = [1 - (b + d)\Delta t]P(n, t) + b\Delta t \sum_{m=0}^{n} P(m, t)P(n - m, t) + d\Delta t \delta_{n,0} . \tag{2}
\]

- In the \( \Delta t \to 0 \) we have

\[
\frac{\partial P(n, t)}{\partial t} = -(b + d)P(n, t) + b \sum_{m=0}^{n} P(m, t)P(n - m, t) + d \delta_{n,0} . \tag{3}
\]

- We can solve this using standard generating functions.
Number of Particles in the system

The solutions are

\[ P(0, t) = \frac{d(e^{bt} - e^{dt})}{be^{bt} - de^{dt}}, \quad P(n \geq 1, t) = (b - d)^2 e^{(b+d)t} \frac{b^{n-1}(e^{bt} - e^{dt})^{n-1}}{(be^{bt} - de^{dt})^{n+1}}. \] (4)

In the critical regime \((b = d)\) this reduces to

\[ P(0, t) = \frac{bt}{1 + bt}, \quad P(n \geq 1, t) = \frac{(bt)^{n-1}}{(1 + bt)^{n+1}}. \] (5)

The average number of particles is

\[ \langle N(t) \rangle = e^{(b-d)t}. \] (6)
The Rightmost Particle

- \( C(n, x, t) \) = joint probability that there are \( n \) particles in the system at time \( t \) with all the particles to the left of \( x \).

- Conditional Probability \( Q(x, t|n) = \frac{C(n, x, t)}{P(n, t)} \)

- PDF of the position of the rightmost particle

\[
P(x, t|n) = \frac{\partial}{\partial x} Q(x, t|n). \tag{7}
\]

- The initial condition is

\[
Q(x, 0|n) = \theta(x) \quad \text{for} \quad n > 1 \tag{8}
\]

- The boundary conditions are

\[
Q(x, t|n) = \begin{cases} 
1 & \text{for} \quad x \to \infty \\
0 & \text{for} \quad x \to -\infty.
\end{cases} \tag{9}
\]
We use the backward Fokker Planck approach.

We have

\[
C(n, x, t + \Delta t) = (1 - (b + d)\Delta t) \langle C(n, x - \eta(0)\Delta t, t) \rangle_{\eta(0)}
\]

\[
+ b\Delta t \sum_{r=0}^{n} C(r, x, t)C(n - r, x, t) + d\Delta t \delta_{n,0} . \tag{10}
\]

In the \( \Delta t \to 0 \) we have

\[
\frac{\partial C(n, x, t)}{\partial t} = D \frac{\partial^2 C(n, x, t)}{\partial x^2} - (b + d)C(n, x, t) + \\
2bP(0, t)C(n, x, t) + b \sum_{r=1}^{n-1} C(r, x, t)C(n - r, x, t) + d \delta_{n,0} . \tag{11}
\]

Linear equation which can be solved recursively.
Relation to FKPP Equation

- For *unconditioned* BBM: \( F(x, t) = \sum_{n=0}^{\infty} C(n, x, t) \). One recovers
  \[
  \frac{\partial F(x, t)}{\partial t} = D \frac{\partial^2 F(x, t)}{\partial x^2} - (b + d)F(x, t) + bF^2(x, t) + d ,
  \]  
  \( (12) \)

- For \( b > d \): *Fisher-Kolmogorov-Petrovsky-Piscounov* type of non-linear equations which allow for a traveling front solution with a well defined front velocity \( v \).

- For \( b = d \): the solution is diffusive at late times (the non-linearities give rise to only sub-leading corrections).

- Unfortunately, for finite \( t \), this is **not exactly solvable**.
Late Time Behaviour

We can remove the linear term by making the transformation

$$C(n, x, t) = e^{\int f(t') dt'} C^\circ(n, x, t) = \frac{e^{(b+d)t}}{(be^{bt} - de^{dt})^2} C^\circ(n, x, t). \quad (13)$$

with

$$f(t) = 2bP(0, t) - (b + d) = (d - b) \frac{be^{bt} + de^{dt}}{be^{bt} - de^{dt}}. \quad (14)$$

We then have

$$\frac{\partial C^\circ(n, x, t)}{\partial t} = D \frac{\partial^2 C^\circ(n, x, t)}{\partial x^2}$$

$$+ \frac{be^{(b+d)t}}{(be^{bt} - de^{dt})^2} \sum_{r=1}^{n-1} C^\circ(r, x, t) C^\circ(n - r, x, t) \quad (15)$$
For the conditional probability \( Q(x, t|n) \) we have

\[
\frac{\partial Q(x, t|n)}{\partial t} = D \frac{\partial^2 Q(x, t|n)}{\partial x^2} + \frac{(b - d)^2 e^{(b+d)t}}{(e^{bt} - e^{dt})(be^{bt} - de^{dt})} \sum_{r=1}^{n-1} \left[ Q(x, t|r)Q(x, t|n-r) - Q(x, t|n) \right]. \tag{16}
\]

By conditioning on \( n \) we obtain a set of linear diffusion equations with source terms which can be solved recursively starting from \( n = 1 \), for all \( t, b \) and \( d \).
The general diffusion equation with a time-dependent source term

\[
\frac{\partial}{\partial t} G(x, t) = D \frac{\partial^2}{\partial x^2} G(x, t) + \sigma(x, t), \quad (17)
\]

With a given initial condition \( G(x, 0) \),

Has the exact solution

\[
G(x, t) = \int_{-\infty}^{\infty} \frac{dx'}{\sqrt{4\pi D t}} \exp \left( -\frac{(x - x')^2}{4Dt} \right) G(x', 0) \\
+ \int_0^t \frac{dt'}{\sqrt{4\pi D(t - t')}} \int_{-\infty}^{\infty} dx' \exp \left( -\frac{(x - x')^2}{4D(t - t')} \right) \sigma(x', t') . \quad (18)
\]
Small n solutions

- For \( n = 1 \) (no source term) we have the exact solution

\[
Q(x, t|1) = \frac{1}{2} \text{erfc} \left( \frac{-x}{\sqrt{4Dt}} \right),
\]

where \( \text{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_{x}^{\infty} e^{-u^2} \, du \) is the complementary error function.

- The corresponding PDF of the position of the rightmost particle is

\[
P(x, t|1) = \frac{\partial}{\partial x} Q(x, t|1) = \frac{1}{\sqrt{4\pi Dt}} \exp \left( -\frac{x^2}{4Dt} \right).
\]

- This is purely diffusive at all times.

- For \( n = 2 \) we have

\[
Q(x, t|2) = (b - d)^2 \left( \frac{be^{bt} - de^{dt}}{e^{bt} - e^{dt}} \right) \int_{0}^{t} \frac{dt'}{\sqrt{4\pi D(t - t')}} \times
\]

\[
\frac{e^{(b+d)t'}}{(be^{bt'} - de^{dt'})^2} \exp \left( -\frac{(x - x')^2}{4D(t - t')} \right) \frac{1}{4} \text{erfc}^2 \left( -\frac{x'}{\sqrt{4Dt'}} \right).
\]
At late times:

\[ Q(x, t|2) \rightarrow \frac{1}{2} \text{erfc} \left( -\frac{x}{\sqrt{4Dt}} \right). \]  

(22)
The cumulative probability is bounded for all $x$ and $t$ $(0 < Q(x, t|n) < 1)$.

Therefore at large $t$, the source term tends to zero as $\sim e^{-|b-d|t}$ (for $b \neq d$), and $\sim 1/(bt^2)$ (for $b = d$).

Thus, at large times $Q(x, t|n)$ obeys the simple diffusion equation for all $n \geq 1$ and the solution behaves for large $t$ as

$$Q(x, t|n) \sim \frac{1}{2} \text{erfc} \left( \frac{-x}{\sqrt{4Dt}} \right),$$

independently of $n$.

The PDF of the rightmost (and by symmetry leftmost) particle is diffusive at large times.
Interpretation

- Conditioning **slows down the motion of the rightmost particle** from ballistic to diffusive.

- For $b > d$ one picks up contributions **only from atypical diffusive trajectories**. $n_{typical} \approx e^{(b-d)t}$.

- For $b \leq d$, this **correctly describes** the late time behavior of the system. $n_{typical} \approx bt$.

- Although the individual behaviour of the particles is diffusive, they are **strongly correlated**.

- In order to understand these correlations, we study **the gaps between the successive particles**.
Remarkably (as we show), the PDFs of these gaps become stationary at large times.

We focus on the first gap $g_1(t) = x_1(t) - x_2(t)$.

We define $P(n, x_1, x_2, t) = \text{PDF that there are exactly } n \text{ particles } (n \geq 2) \text{ at time } t$, with the first particle at position $x_1$ and the second at position $x_2 < x_1$.

We start with the simplest case $n = 2$ which is already nontrivial.
Using the Backward Fokker-Planck approach

\[
P(2, x_1, x_2, t + \Delta t) = \]
\[
(1 - (b + d)\Delta t) \langle P(2, x_1 - \eta(0)\Delta t, x_2 - \eta(0)\Delta t, t) \rangle_{\eta(0)} + 2b\Delta t P(0, t)P(2, x_1, x_2, t) + 2b\Delta t P(1, x_1, t)P(1, x_2, t). \tag{24}
\]
Expanding and taking the limit $\Delta t \to 0$, we have

\[
\frac{\partial}{\partial t} P(2, x_1, x_2, t) = D \left( \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} \right)^2 P(2, x_1, x_2, t) + f(t) P(2, x_1, x_2, t) + 2bP(1, x_1, t)P(1, x_2, t) , \quad (25)
\]
Exact Solution

- **We remove the linear term** by the customary transformation

\[ P(2, x_1, x_2, t) = \frac{e^{(b+d)t}}{(be^{bt} - de^{dt})^2} P^o(2, x_1, x_2, t). \]  

(26)

- We then have:

\[
\frac{\partial}{\partial t} P^o(2, x_1, x_2, t) = D \left( \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} \right)^2 P^o(2, x_1, x_2, t) \\
+ 2b \frac{(be^{bt} - de^{dt})^2}{e^{(b+d)t}} P(1, x_1, t)P(1, x_2, t). 
\]  

(27)

- Change of variables (to **Centre of Mass** and **Gap**)

\[
s = \frac{x_1 + x_2}{2} \\
g_1 = x_1 - x_2 > 0
\]  

(28)
This yields

\[
\frac{\partial}{\partial t} P^o(2, s, g_1, t) = D \left( \frac{\partial}{\partial s} \right)^2 P^o(2, s, g_1, t) \\
+ 2b \frac{e^{(b+d)t}}{(be^{bt} - de^{dt})^2} (b - d)^4 \frac{1}{4\pi Dt} \exp \left( - \frac{2s^2 + \frac{1}{2}g_1^2}{4Dt} \right) .
\]  

(29)

Which is a diffusion equation with a source term!
Exact Solution (Cont.)

- Conditional PDF \( P(s, g_1, t|2) = \frac{P(2,s,g_1,t)}{P(2,t)} \).

- We have

\[
P(s, g_1, t|2) = \left( \frac{be^{bt} - de^{dt}}{b(b - d)^2(e^{bt} - e^{dt})} \right) P^o(2, s, g_1, t) .
\]

(30)

- Integrating w.r.t. to \( s' \) we have the exact solution:

\[
P(s, g_1, t|2) = \frac{(b - d)^2}{2\pi D} \left( \frac{be^{bt} - de^{dt}}{e^{bt} - e^{dt}} \right) \times
\]

\[
\int_0^t dt' \frac{e^{(b+d)t'}}{(be^{bt'} - de^{dt'})^2} \frac{e^{-\frac{g_1^2}{8Dt'}} - \frac{s^2}{2D(2t-t')}}{\sqrt{t'(2t - t')}} .
\]

(31)
Marginal Distribution of the Centre of Mass

- Given the exact solution we can derive the **marginal distributions** of $s$ and $g_1$ respectively.

- **Integrating over** $g_1$ gives us the marginal PDF of the centre of mass

$$P(s, t|2) = (b - d)^2 \left( \frac{be^{bt} - de^{dt}}{e^{bt} - e^{dt}} \right) \times$$

$$
\int_0^t dt' \frac{e^{(b+d)t'}}{(be^{bt'} - de^{dt'})^2} \exp\left(-\frac{s^2}{2D(2t-t')}ight) \frac{\sqrt{\frac{2}{\pi}}}{\sqrt{2\pi D(2t-t')}}.
$$

- This is **dominated by the region** $t' \to 0$, leading to

$$P(s, t|2) \sim \frac{1}{\sqrt{4\pi Dt}} \exp\left(-\frac{s^2}{4Dt}\right)$$

for large $t$, consistent with diffusive behaviour.
Marginal Distribution of the Gap

- **Integrating over the centre of mass** variable $s$ marginal PDF of the gap

\[
P(g_1, t|2) = (b - d)^2 \left( \frac{b e^{bt} - de^{dt}}{e^{bt} - e^{dt}} \right) \int_0^t dt' \frac{e^{(b+d)t'}}{(be^{bt'} - de^{dt'})^2} \exp\left(-\frac{g_1^2}{8Dt'}\right) \sqrt{\frac{2\pi}{2Dt'}}.
\] (33)

- This gap distribution becomes **stationary at large times**

\[
P(g_1, t \to \infty|2) = p(g_1|2)
\]

- We have

\[
p(g_1|2) = (b - d)^2 \max(b, d) \int_0^\infty dt' \frac{e^{(b+d)t'}}{(be^{bt'} - de^{dt'})^2} \exp\left(-\frac{g_1^2}{8Dt'}\right) \sqrt{2\pi Dt'}.
\] (34)
Stationary Behaviour

- This stationary gap PDF has the following **asymptotic behaviour** for $g_1 \gg 1$

\[
p(g_1|2) \sim \begin{cases} 
\frac{|b - d|^{3/2}}{\sqrt{2D} \max(b, d)} \exp \left(-\sqrt{\frac{|b - d|}{2D}} g_1 \right), & \text{for } b \neq d, \\
8 \left(\frac{D}{b}\right) g_1^{-3}, & \text{for } b = d.
\end{cases}
\]

- **Exponential decay** in the off-critical phases.

- **Scale-free power law decay at the critical point.**
Higher n Sectors

- For any $n > 2$, following the same procedure:

$$\frac{\partial P(n, x_1, x_2, t)}{\partial t} = D \left( \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} \right)^2 P(n, x_1, x_2, t) + f(t)P(n, x_1, x_2, t) + b S(n, x_1, x_2, t), \quad (35)$$

- The source term is:

$$S(n, x_1, x_2, t) = \int_{-\infty}^{x_2} dx_3 \left[ 2 \sum_{\tau \in S_3} P(1, x_{\tau 1}, t)P(n - 1, x_{\tau 2}, x_{\tau 3}, t) \right.$$  
$$+ \sum_{r=2}^{n-2} \int_{-\infty}^{x_3} dx_4 \sum_{\tau \in S_4} P(r, x_{\tau 1}, x_{\tau 2}, t)P(n - r, x_{\tau 3}, x_{\tau 4}, t) \right], \quad (36)$$

- Once again, the gap PDF becomes stationary at large times, $P(g_1, t \to \infty | n) \to p(g_1 | n)$. 
Asymptotic Behaviour

- Leading contribution to $S$ for $g_1 = x_1 - x_2 \gg 1$ arises from the term
  \[ 2b \ P(1, x_1, t) \int_{-\infty}^{x_2} dx_3 \ P(n - 1, x_2, x_3, t) = 2b \ P(1, x_1, t)P(n - 1, x_2, t), \]  
  \hspace{1cm} (37) 

- Rightmost particle is diffusive at large $t$: $P(n - 1, x_2, t) \sim P(1, x_2, t)$,

- Therefore for large $t$
  \[ 2b \ P(1, x_1, t) \int_{-\infty}^{x_2} dx_3 \ P(n - 1, x_2, x_3, t) \sim 2b \ P(1, x_1, t)P(1, x_2, t), \]  
  \hspace{1cm} (38) 

- This is precisely the source term for the two-particle case, leading to $p(g_1|n) \sim p(g_1|2)$ independently of $n \geq 2$.

- All other terms in $S$ involve a large gap between particles generated by the same offspring and are suppressed.
Similar arguments show $p(g_k = x_k - x_{k+1} | n) \sim p(g_1 | 2)$ for $g_k \gg 1$.

**Figure:** Dominant terms contributing to the large gap behaviour for a) the first gap $g_1(t)$ and c) the $k$-th gap $g_k(t)$. Figure b) shows a realization where the large gap is generated by the particles of the same offspring process and is hence suppressed.
Monte Carlo Simulations

- Simulations in the **off-critical** regime

\[ P(g_1; t | 2) \]

\[ P(g_1; t | 3) \]

**Figure:** Two and Three particle Sectors
Monte Carlo Simulations (Cont.)

- Simulations in the **critical regime**

Figure: Two particle sector
Monte Carlo Simulations (Cont.)

- Simulations in the **critical regime**

**Figure:** Three particle sector
Universality

\[ S(g_k, t|n) \]

2 particles
3 particles
4 particles
5 particles
6 particles
7 particles
8 particles

1st gap
2nd gap
3rd gap
4th gap
5th gap
Conclusion

- We obtained exact **analytical results for the gap statistics** of the extreme particles of BBM.

- This was possible by **conditioning on the number of particles in the system**.

- This allowed us to express these evolution equations as a **system of linear diffusion equations with source terms**, which we could then solve **recursively**.

- We generalized this procedure for all particle sectors and showed that the stationary gap distributions have **universal tails**.

- It will be interesting to extend our analysis to the question of **$k$-point correlation functions** of this process.
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