# Correlated Extreme Values in Branching Brownian Motion

Kabir Ramola

Laboratoire de Physique Théorique et Modèles Statistiques, Orsay, France

In collaboration with Satya N. Majumdar and Grégory Schehr

Monday $5^{\text{th}}$ January, 2015

# References

Kabir Ramola, Satya N. Majumdar, Grégory Schehr:

- *Universal Order and Gap Statistics of Critical Branching Brownian Motion*, Phys. Rev. Lett. **112**, 210602, (2014).
- *Branching Brownian Motion Conditioned on Particle Numbers*, Chaos, Solitons & Fractals (special edition on Extreme Value Statistics), (2015).

# Introduction

- Branching processes are prototypical models of systems where **new particles are generated at every time step**.

- Well studied in the context of **evolution, epidemic spreads, nuclear reactions** amongst others.

- Related to several models such as continuum limit of **branching-annihilating-random-walk** (DP Universality), **GREM**.

- Used in the modelling of **disordered systems and spin-glasses** where energy levels are random variables.

# Branching Brownian Motion

At each time step $[t, t + \Delta t]$ the particle can:

- **A)** die with probability $d\Delta t$
- **B)** split into two independent particles with probability $b\Delta t$
- **C)** diffuse by a distance $\Delta x = \eta(t)\Delta t$, with probability $1 - (b + d)\Delta t$.

$$\langle \eta(t) \rangle = 0, \quad \langle \eta(t_1)\eta(t_2) \rangle = 2D\delta(t_1 - t_2) \tag{1}$$

# Branching Brownian Motion

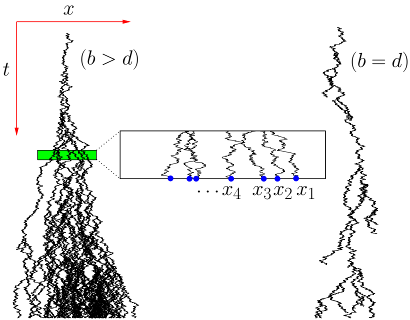


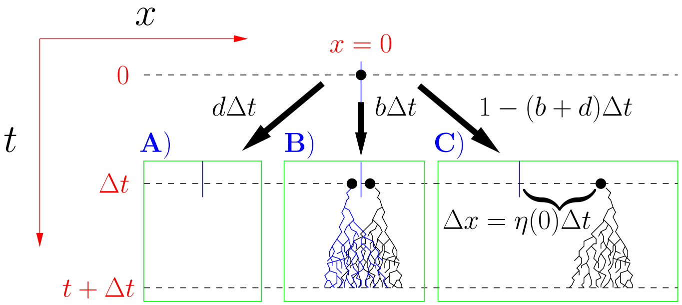

Figure: A realization of the dynamics of branching Brownian motion with death (left) in the supercritical regime ($b > d$) and (right) in the critical regime ($b = d$). The particles are numbered sequentially from right to left as shown in the inset.

# Extreme Value Statistics

- Extreme value statistics has been growing in prominence.
- In many real world examples the extreme value is not independent of the rest of the set and there are **strong correlations between near-extreme values**.
- Examples include **extreme temperatures** as part of heat or cold waves, **earthquakes and financial crashes** where extreme fluctuations are accompanied by foreshocks and aftershocks.
- Particularly important in **disordered systems** where energy levels near the ground state become important at low but finite temperature.
- Although EVS of independent identically distributed (i.i.d.) variables are fully understood, **very few analytical results for strongly correlated random variables**.

# The Backward Fokker-Planck Approach

- We look at the contribution from the **first time step** $[0, \Delta t]$ to the final time step $t + \Delta t$



# Number of Particles in the system

- $P(n,t) =$ Probability there are exactly $n$ particles at time $t$.
- Using the Backward Fokker-Planck approach

$$\begin{aligned} P(n, t+\Delta t) &= [1-(b+d)\Delta t]P(n,t) + \\ & b\Delta t \sum_{m=0}^{n} P(m,t)P(n-m,t) + d\Delta t\, \delta_{n,0}\,. \end{aligned} \tag{2}$$

- In the $\Delta t \to 0$ we have

$$\frac{\partial P(n,t)}{\partial t} = -(b+d)P(n,t) + b\sum_{m=0}^{n} P(m,t)P(n-m,t) + d\,\delta_{n,0}\,. \tag{3}$$

- We can solve this using standard **generating functions**.

# Number of Particles in the system

- The solutions are

$$P(0,t) = \frac{d(e^{bt} - e^{dt})}{be^{bt} - de^{dt}}\ , \qquad P(n \geq 1, t) = (b-d)^2 e^{(b+d)t} \frac{b^{n-1}(e^{bt} - e^{dt})^{n-1}}{(be^{bt} - de^{dt})^{n+1}}\ . \tag{4}$$

- In the critical regime $(b = d)$ this reduces to

$$P(0,t) = \frac{bt}{1+bt}, \qquad P(n \geq 1, t) = \frac{(bt)^{n-1}}{(1+bt)^{n+1}}\ . \tag{5}$$

- The average number of particles is

$$\langle N(t) \rangle = e^{(b-d)t}. \tag{6}$$

# The Rightmost Particle

- $C(n, x, t) =$ joint probability that there are $n$ particles in the system at time $t$ with all the particles **to the left of** $x$.

- Conditional Probability $Q(x, t|n) = \frac{C(n,x,t)}{P(n,t)}$

- PDF of the position of the rightmost particle

$$P(x, t|n) = \frac{\partial}{\partial x} Q(x, t|n). \tag{7}$$

- The initial condition is

$$Q(x, 0|n) = \theta(x) \quad \text{for} \quad n > 1 \tag{8}$$

- The boundary conditions are

$$Q(x, t|n) = \begin{cases} 1 & \text{for} \quad x \to \infty \\ 0 & \text{for} \quad x \to -\infty. \end{cases} \tag{9}$$

# The Rightmost Particle (Cont.)

- We use the backward Fokker Planck approach.
- We have

$$C(n,x,t+\Delta t) = (1-(b+d)\Delta t)\,\langle C(n,x-\eta(0)\Delta t,t)\rangle_{\eta(0)} + b\Delta t\sum_{r=0}^{n} C(r,x,t)C(n-r,x,t) + d\Delta t\,\delta_{n,0}\ . \quad (10)$$

- In the $\Delta t \to 0$ we have

$$\frac{\partial C(n,x,t)}{\partial t} = D\frac{\partial^2 C(n,x,t)}{\partial x^2} - (b+d)C(n,x,t) + 2bP(0,t)C(n,x,t) + b\sum_{r=1}^{n-1} C(r,x,t)C(n-r,x,t) + d\,\delta_{n,0}\ . \quad (11)$$

- Linear equation which **can be solved recursively**.

# Relation to FKPP Equation

- For *unconditioned* BBM: $F(x,t) = \sum_{n=0}^{\infty} C(n,x,t)$. One recovers

$$\frac{\partial F(x,t)}{\partial t} = D\frac{\partial^2 F(x,t)}{\partial x^2} - (b+d)F(x,t) + bF^2(x,t) + d \ , \qquad (12)$$

- For $b > d$: Fisher-Kolmogorov-Petrovsky-Piscounov type of **non-linear equations which allow for a traveling front solution** with a well defined front velocity $v$.

- For $b = d$: the solution is diffusive at late times (the non-linearities give rise to only sub-leading corrections).

- Unfortunately, for finite $t$, this is **not exactly solvable**.

# Late Time Behaviour

- We can remove the linear term by making the transformation

$$C(n,x,t) = e^{\int f(t')dt'} C^{\circ}(n,x,t) = \frac{e^{(b+d)t}}{(be^{bt} - de^{dt})^2} C^{\circ}(n,x,t) \,. \quad (13)$$

with

$$f(t) = 2bP(0,t) - (b+d) = (d-b)\frac{be^{bt} + de^{dt}}{be^{bt} - de^{dt}} \,. \quad (14)$$

- We then have

$$\begin{aligned} \frac{\partial C^{\circ}(n,x,t)}{\partial t} &= D\frac{\partial^2 C^{\circ}(n,x,t)}{\partial x^2} \\ &\quad + \frac{be^{(b+d)t}}{(be^{bt} - de^{dt})^2} \sum_{r=1}^{n-1} C^{\circ}(r,x,t) C^{\circ}(n-r,x,t) \quad (15) \end{aligned}$$

# Late Time Behaviour (Cont.)

- For the conditional probability $Q(x,t|n)$ we have

$$\frac{\partial Q(x,t|n)}{\partial t} = D\frac{\partial^2 Q(x,t|n)}{\partial x^2} + \frac{(b-d)^2 e^{(b+d)t}}{(e^{bt}-e^{dt})(be^{bt}-de^{dt})}\sum_{r=1}^{n-1}\Big[Q(x,t|r)Q(x,t|n-r) - Q(x,t|n)\Big]. \quad (16)$$

- By conditioning on $n$ we obtain a **set of linear diffusion equations with source terms** which can be solved **recursively starting from $n=1$, for all** $t$, $b$ and $d$.

# Diffusion Equation with a Source

- The general diffusion equation with a time-dependent source term

$$\frac{\partial}{\partial t}G(x,t) = D\frac{\partial^2}{\partial x^2}G(x,t) + \sigma(x,t), \tag{17}$$

- With a given initial condition $G(x,0)$,
- Has the exact solution

$$G(x,t) = \int_{-\infty}^{\infty} \frac{dx'}{\sqrt{4\pi Dt}} \exp\left(-\frac{(x-x')^2}{4Dt}\right) G(x',0)$$
$$+ \int_0^t \frac{dt'}{\sqrt{4\pi D(t-t')}} \int_{-\infty}^{\infty} dx' \exp\left(-\frac{(x-x')^2}{4D(t-t')}\right) \sigma(x',t') \ . \tag{18}$$

# Small n solutions

- For $n = 1$ **(no source term)** we have the exact solution

$$Q(x,t|1) = \frac{1}{2}\mathrm{erfc}\left(\frac{-x}{\sqrt{4Dt}}\right), \tag{19}$$

where $\mathrm{erfc}(x) = \frac{2}{\sqrt{\pi}} \int_x^\infty e^{-u^2}\, du$ is the complementary error function.

- The corresponding PDF of the position of the rightmost particle is

$$P(x,t|1) = \frac{\partial}{\partial x} Q(x,t|1) = \frac{1}{\sqrt{4\pi Dt}} \exp\left(-\frac{x^2}{4Dt}\right). \tag{20}$$

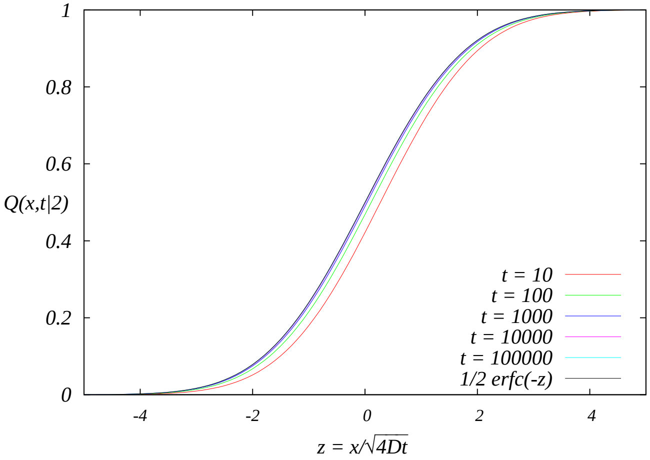
- This is purely diffusive at *all times*.
- For $n = 2$ we have

$$Q(x,t|2) = (b-d)^2 \left(\frac{be^{bt} - de^{dt}}{e^{bt} - e^{dt}}\right) \int_0^t \frac{dt'}{\sqrt{4\pi D(t-t')}} \times \frac{e^{(b+d)t'}}{(be^{bt'} - de^{dt'})^2} \exp\left(-\frac{(x-x')^2}{4D(t-t')}\right) \frac{1}{4}\mathrm{erfc}^2\left(-\frac{x'}{\sqrt{4Dt'}}\right). \tag{21}$$

# Small n solutions (Cont.)

- At late times:

$$Q(x,t|2) \to \frac{1}{2}\mathrm{erfc}\left(-\frac{x}{\sqrt{4Dt}}\right). \qquad (22)$$



# General n

- The cumulative probability **is bounded** for all $x$ and $t$ $(0 < Q(x,t|n) < 1)$.
- Therefore at large $t$, **the source term tends to zero** as $\sim e^{-|b-d|t}$ (for $b \neq d$), and $\sim 1/(bt^2)$ (for $b = d$).
- Thus, at large times $Q(x,t|n)$ **obeys the simple diffusion equation** for all $n \geq 1$ and the solution behaves for large $t$ as

$$Q(x,t|n) \sim \frac{1}{2}\mathrm{erfc}\left(\frac{-x}{\sqrt{4Dt}}\right) \ , \tag{23}$$

  independently of $n$.
- The PDF of the rightmost (and by symmetry leftmost) particle **is diffusive at large times**.

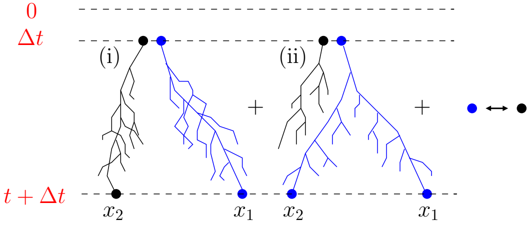
# Interpretation

- Conditioning **slows down the motion of the rightmost particle** from ballistic to diffusive.
- For $b > d$ one picks up contributions **only from atypical diffusive trajectories**. $n_{typical} \approx e^{(b-d)t}$.
- For $b \leq d$, this **correctly describes** the late time behavior of the system. $n_{typical} \approx bt$.
- Although the individual behaviour of the particles is diffusive, **they are strongly correlated**.
- In order to understand these correlations, we study **the gaps between the succesive particles**.

# Gap Statistics

- Remarkably (as we show), the PDFs of these gaps **become stationary at large times**.
- We focus on the first gap $g_1(t) = x_1(t) - x_2(t)$.
- We define $P(n, x_1, x_2, t) =$ PDF that there are exactly $n$ particles $(n \geq 2)$ at time $t$, with the first particle at position $x_1$ and the second at position $x_2 < x_1$.
- We start with **the simplest case** $n = 2$ which is already nontrivial.

# Two Particle Sector




- Using the Backward Fokker-Planck approach

$$P(2, x_1, x_2, t + \Delta t) =$$
$$(1 - (b + d)\Delta t) \langle P(2, x_1 - \eta(0)\Delta t, x_2 - \eta(0)\Delta t, t) \rangle_{\eta(0)}$$
$$+ 2b\Delta t P(0, t) P(2, x_1, x_2, t) + 2b\Delta t P(1, x_1, t) P(1, x_2, t). \quad (24)$$

# Two Particle Sector (Cont.)

- Expanding and taking the limit $\Delta t \to 0$, we have

$$\frac{\partial}{\partial t}P(2, x_1, x_2, t) = D\left(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}\right)^2 P(2, x_1, x_2, t) + f(t)P(2, x_1, x_2, t) + 2bP(1, x_1, t)P(1, x_2, t)\ , \quad (25)$$

# Exact Solution

- We **remove the linear term** by the customary transformation

$$P(2, x_1, x_2, t) = \frac{e^{(b+d)t}}{(be^{bt} - de^{dt})^2} P^{\circ}(2, x_1, x_2, t). \tag{26}$$

- We then have:

$$\frac{\partial}{\partial t} P^{\circ}(2, x_1, x_2, t) = D \left( \frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2} \right)^2 P^{\circ}(2, x_1, x_2, t) + 2b \frac{(be^{bt} - de^{dt})^2}{e^{(b+d)t}} P(1, x_1, t) P(1, x_2, t) \,. \tag{27}$$

- Change of variables (to **Centre of Mass** and **Gap**)

$$s = \frac{x_1 + x_2}{2}$$
$$g_1 = x_1 - x_2 > 0 \tag{28}$$

# Exact Solution (Cont.)

- This yields

$$\frac{\partial}{\partial t} P^{\circ}(2, s, g_1, t) = D \left( \frac{\partial}{\partial s} \right)^2 P^{\circ}(2, s, g_1, t)$$
$$+ 2b \frac{e^{(b+d)t}}{(be^{bt} - de^{dt})^2} (b-d)^4 \frac{1}{4\pi D t} \exp\left( -\frac{2s^2 + \frac{1}{2} g_1^2}{4Dt} \right). \tag{29}$$

- Which is a **diffusion equation with a source term**!

# Exact Solution (Cont.)

- Conditional PDF $P(s, g_1, t|2) = \frac{P(2,s,g_1,t)}{P(2,t)}$.
- We have

$$P(s, g_1, t|2) = \left( \frac{be^{bt} - de^{dt}}{b(b-d)^2(e^{bt} - e^{dt})} \right) P^{\circ}(2, s, g_1, t) \,. \qquad (30)$$

- Integrating w.r.t. to $s'$ we have the exact solution:

$$P(s, g_1, t|2) = \frac{(b-d)^2}{2\pi D} \left( \frac{be^{bt} - de^{dt}}{e^{bt} - e^{dt}} \right) \times$$
$$\int_0^t dt' \frac{e^{(b+d)t'}}{(be^{bt'} - de^{dt'})^2} \frac{e^{-\frac{g_1^2}{8Dt'} - \frac{s^2}{2D(2t-t')}}}{\sqrt{t'(2t-t')}}. \qquad (31)$$

# Marginal Distribution of the Centre of Mass

- Given the exact solution we can derive the **marginal distributions** of $s$ and $g_1$ respectively.
- **Integrating over** $g_1$ gives us the marginal PDF of the centre of mass

$$P(s,t|2) = (b-d)^2 \left( \frac{be^{bt} - de^{dt}}{e^{bt} - e^{dt}} \right) \times \int_0^t dt' \frac{e^{(b+d)t'}}{(be^{bt'} - de^{dt'})^2} \frac{\exp(-\frac{s^2}{2D(2t-t')})}{\sqrt{2\pi D(2t-t')}} \,. \tag{32}$$

- This is **dominated by the region** $t' \to 0$, leading to $P(s,t|2) \sim \frac{1}{\sqrt{4\pi Dt}} \exp\left(-\frac{s^2}{4Dt}\right)$ for large $t$, consistent with diffusive behaviour.

# Marginal Distribution of the Gap

- **Integrating over the centre of mass** variable $s$ marginal PDF of the gap

$$P(g_1, t|2) = (b-d)^2 \left( \frac{be^{bt} - de^{dt}}{e^{bt} - e^{dt}} \right) \int_0^t dt' \frac{e^{(b+d)t'}}{(be^{bt'} - de^{dt'})^2} \frac{\exp(-\frac{g_1^2}{8Dt'})}{\sqrt{2\pi Dt'}} \,. \tag{33}$$

- This gap distribution becomes **stationary at large times** $P(g_1, t \to \infty|2) = p(g_1|2)$
- We have

$$p(g_1|2) = (b-d)^2 \max(b,d) \int_0^\infty dt' \frac{e^{(b+d)t'}}{(be^{bt'} - de^{dt'})^2} \frac{\exp(-\frac{g_1^2}{8Dt'})}{\sqrt{2\pi Dt'}}. \tag{34}$$

# Stationary Behaviour

- This stationary gap PDF has the following **asymptotic behaviour** for $g_1 \gg 1$

$$p(g_1|2) \sim \begin{cases} \frac{|b-d|^{3/2}}{\sqrt{2D}\max(b,d)} \exp\left(-\sqrt{\frac{|b-d|}{2D}}\; g_1\right) \;, & \text{for} \quad b \neq d \;, \\ 8\left(\frac{D}{b}\right) g_1^{-3} \;, & \text{for} \quad b = d \;. \end{cases}$$

- **Exponential decay** in the off-critical phases.

- Scale-free **power law decay at the critical point**.

# Higher n Sectors

- For any $n > 2$, following the same procedure:

$$\frac{\partial P(n, x_1, x_2, t)}{\partial t} = D\left(\frac{\partial}{\partial x_1} + \frac{\partial}{\partial x_2}\right)^2 P(n, x_1, x_2, t) \\ + f(t)P(n, x_1, x_2, t) + b\,\mathcal{S}(n, x_1, x_2, t), \quad (35)$$

- The source term is:

$$\mathcal{S}(n, x_1, x_2, t) = \int_{-\infty}^{x_2} dx_3 \Big[ 2 \sum_{\tau \in S_3} P(1, x_{\tau_1}, t) P(n-1, x_{\tau_2}, x_{\tau_3}, t) \\ + \sum_{r=2}^{n-2} \int_{-\infty}^{x_3} dx_4 \sum_{\tau \in S_4} P(r, x_{\tau_1}, x_{\tau_2}, t) P(n-r, x_{\tau_3}, x_{\tau_4}, t) \Big], \quad (36)$$

- Once again, the gap PDF **becomes stationary** at large times, $P(g_1, t \to \infty | n) \to p(g_1 | n)$.

# Asymptotic Behaviour

- Leading contribution to $\mathcal{S}$ for $g_1 = x_1 - x_2 \gg 1$ arises from the term

$$2b\, P(1, x_1, t) \int_{-\infty}^{x_2} dx_3 P(n-1, x_2, x_3, t) = 2b\, P(1, x_1, t) P(n-1, x_2, t)\,, \quad (37)$$

- Rightmost particle is diffusive at large $t$: $P(n-1, x_2, t) \sim P(1, x_2, t)$,
- Therefore for large $t$

$$2b\, P(1, x_1, t) \int_{-\infty}^{x_2} dx_3 P(n-1, x_2, x_3, t) \sim 2b\, P(1, x_1, t) P(1, x_2, t)\,, \quad (38)$$

- **This is precisely the source term for the two-particle case**, leading to $p(g_1|n) \sim p(g_1|2)$ independently of $n \geq 2$.

- All other terms in $\mathcal{S}$ involve a large gap between particles generated by the same offspring and **are suppressed**.

# Asymptotic Behaviour (Cont.)

- Similar arguments show $p(g_k = x_k - x_{k+1}|n) \sim p(g_1|2)$ for $g_k \gg 1$

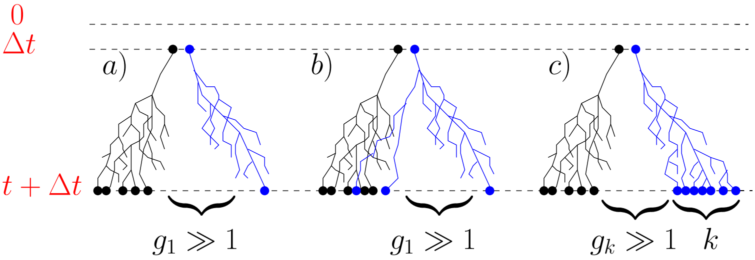



Figure: Dominant terms contributing to the large gap behaviour for a) the first gap $g_1(t)$ and c) the $k$-th gap $g_k(t)$. Figure b) shows a realization where the large gap is generated by the particles of the same offspring process and is hence suppressed.

# Monte Carlo Simulations

- Simulations in the **off-critical regime**

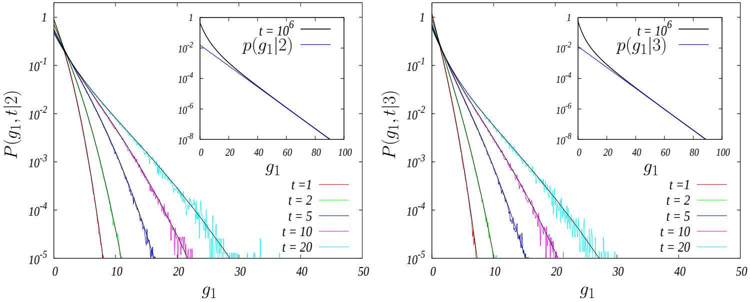



Figure: Two and Three particle Sectors

- Simulations in the **critical regime**

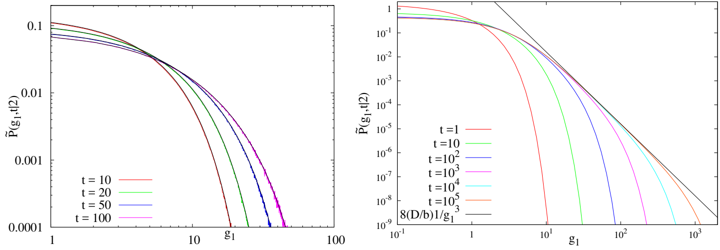



Figure: Two particle sector

# Monte Carlo Simulations (Cont.)

- Simulations in the **critical regime**

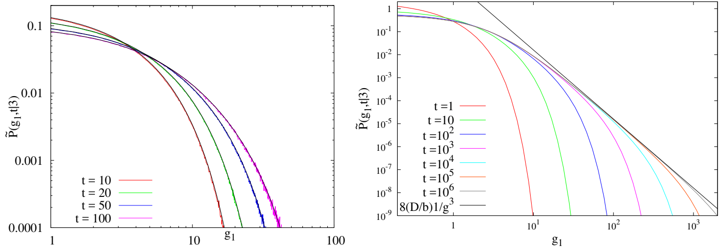
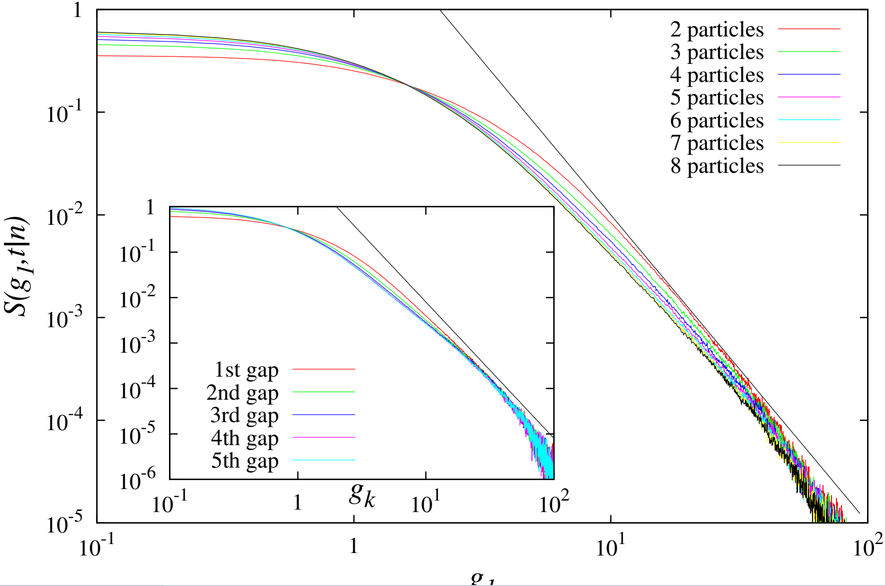



Figure: Three particle sector

# Universality



# Conclusion

- We obtained exact **analytical results for the gap statistics** of the extreme particles of BBM.
- This was possible by **conditioning on the number of particles in the system**.
- This allowed us to express these evolution equations as a **system of linear diffusion equations with source terms**, which we could then solve **recursively**.
- We generalized this procedure for all particle sectors and showed that the stationary gap distributions have **universal tails**.
- It will be interesting to extend our analysis to the question of $k$-**point correlation functions** of this process.

# Acknowledgements

A. Kundu, S. Gupta, A. Gudyma, C. Texier, B. Derrida.

Thank You.