JOURNAL CLUB

FRACTON-ELASTICITY DUALITY

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CLASSICAL ELECTROMAGNETISM

- *Classical Electromagnetism is described by Maxwell's Equations*

- *Governed by the Lagrangian*

$$L = \int d^2x dt(-\tfrac{1}{4}F_{\mu\nu}F^{\mu\nu} + \tfrac{1}{c}j_\mu A^\mu)$$

$$F^{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$$

$$H = \tfrac{1}{2}(E^2 + B^2)$$

- *Theory has charge conservation*

- *What are the analogous equations for Tensor fields?*

TENSOR ELECTROMAGNETISM

- ***Consider a general Gauss's law***

$$\partial_i \partial_j E^{ij} = \rho$$

- ***For a point charge***

$$\partial_i \partial_j E^{ij} = q\delta^{(3)}(r)$$

- ***Generalized Coulomb potential (from dimensional analysis)***

$$E^{ij} = q\left(\alpha \frac{\delta^{ij}}{r} + \beta \frac{r^i r^j}{r^3}\right)$$

- ***This solves Gauss's law***

$$\partial_i E^{ij} = q(\beta - \alpha)\frac{r^j}{r^3}$$

$$\partial_i \partial_j E^{ij} = 4\pi q(\beta - \alpha)\delta^{(3)}(r)$$

- ***If:*** $\beta - \alpha = 1/4\pi$

TENSOR ELECTROMAGNETISM (continued)

- *We therefore have*

$$E^{ij} = q\left(\alpha\frac{\delta^{ij}}{r} + \left(\alpha + \frac{1}{4\pi}\right)\frac{r^i r^j}{r^3}\right)$$

- *Unlike conventional electromagnetism, the static point charge solution is NOT uniquely specified by Gauss's law and rotational symmetry.*

- *Therefore, in order to further constrain the electric field, we must resort to another of the generalized Maxwell equations.*

ADDITIONAL CONSTRAINTS

- *The magnetic field tensor is*

$$B^{ij} = \epsilon^{iab}\partial_a {A_b}^j$$

- *The equation governing the evolution of the magnetic field is*

$$\partial_t B^{ij} = \epsilon^{iab}\partial_a\partial_t {A_b}^j = \epsilon^{iab}\partial_a {E_b}^j$$

- *Therefore, for magnetostatics we require*

$$\epsilon^{iab}\partial_a {E_b}^j = q(\alpha+\beta)\frac{\epsilon^{ija}r_a}{r^3} = 0$$

- *Leading to* $\beta = -\alpha$
- *Along with* $\beta = \alpha + 1/4\pi$

$$\left.\begin{array}{l}\beta = -\alpha \\ \beta = \alpha + 1/4\pi\end{array}\right\} \quad \alpha = -\frac{1}{8\pi}$$

TWO DIMENSIONAL ELASTICITY

- *Each atom can oscillate only a small distance from its equilibrium position:* u^i

- *The most general low-energy action*

$$S = \int d^2x dt \frac{1}{2}\left((\partial_t u^i)^2 - C^{ijk\ell} u_{ij} u_{k\ell} \right)$$

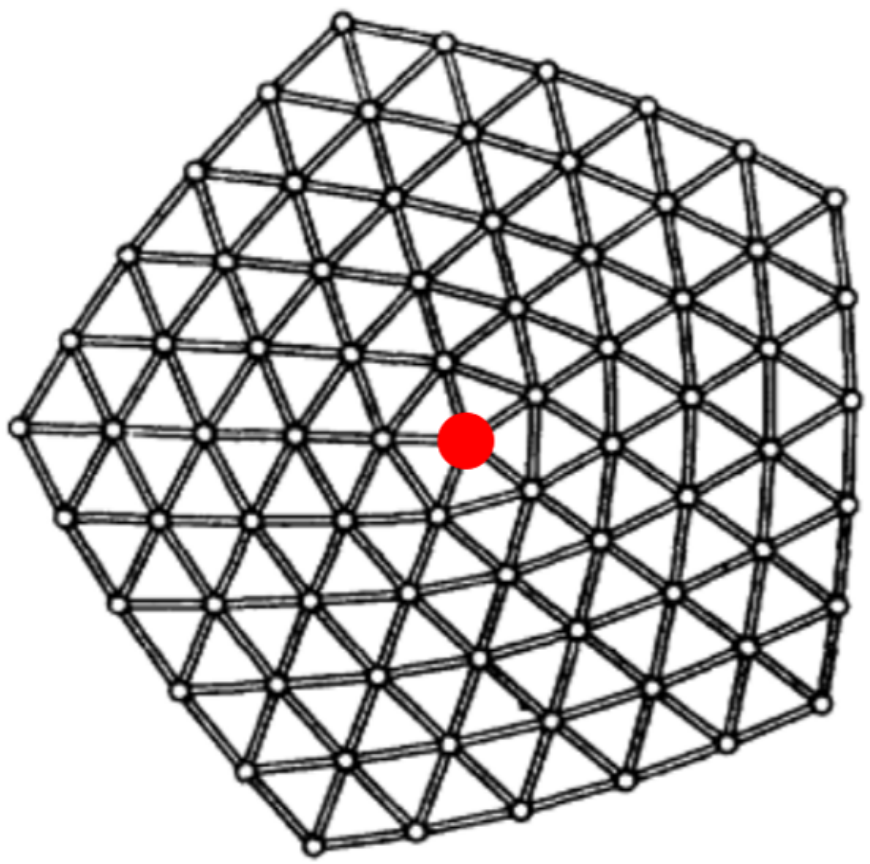
- *Another fundamental quantity is the bond angle*

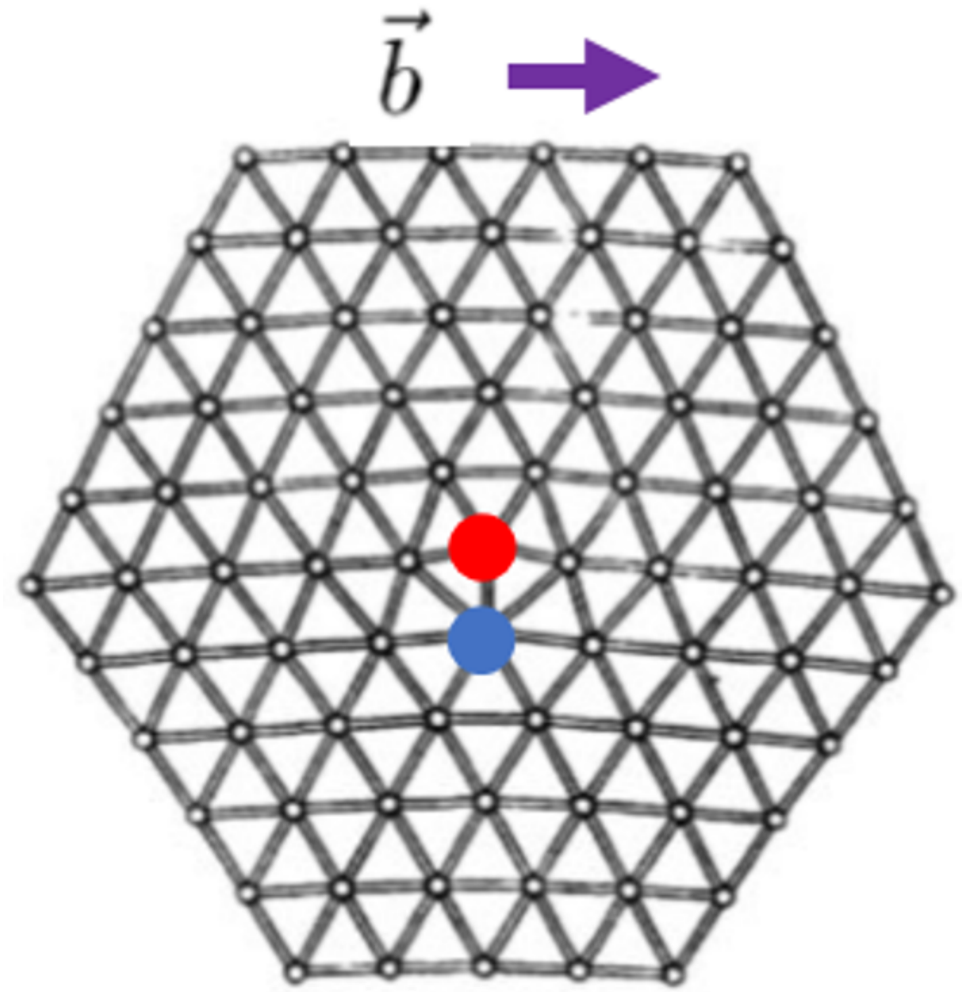
$$\theta_b = \frac{1}{2}\epsilon^{ij}\partial_i u_j$$

- *For an n-fold symmetric crystal*

$$\oint d\ell^i\, \partial_i \theta_b = \frac{2\pi}{n} s$$

DISCLINATIONS AND DISLOCATIONS





TWO DIMENSIONAL ELASTICITY (continued)

- ***The disinclinations in terms of the symmetric strain tensor***

$$\begin{aligned}\frac{2\pi}{n} s &= \oint d\ell^i \, \partial_i \theta_b \\ &= -\int d^2x \, \epsilon^{i\ell} \epsilon^{jk} \partial_\ell \partial_k u_{ij} - \frac{1}{2} \int d^2x \, \epsilon^{i\ell} \partial_\ell (\epsilon^{kj} \partial_k \partial_j u_i),\end{aligned}$$

- ***Define a disclination density***

$$\rho_s = \epsilon^{i\ell} \epsilon^{jk} \partial_\ell \partial_k u_{ij}$$

- ***Total disclination number as:***

$$-\frac{2\pi}{n} s = \int d^2x \left(\rho_s - \epsilon_{i\ell} \partial^i \left(\tfrac{1}{2} \epsilon^{kj} \partial_k \partial_j u_i\right) \right)$$

TWO DIMENSIONAL ELASTICITY (continued)

- *The Burger's vector is given by*

$$\oint d\ell^i \, \partial_i u_j = b_j \, .$$

- *Once again this can be expressed in terms of the strain tensor*

$$b_n = \epsilon_{mn} \int d^2x \, (\rho x^m) = \oint d\ell^i x^m \epsilon_{mn} \epsilon^{jk} \partial_k \partial_i u_j + \int d^2x \, \rho_{b,n}$$

$$= \oint d\ell^i \epsilon_{in} \epsilon^{jk} \partial_k u_j + \oint d\ell^i \partial_i u_n = \oint d\ell^i \partial_i u_n$$

- *Once again this can be expressed in terms of the strain tensor. Define the dislocation density*

$$\rho_b^n = \epsilon^{ik} \partial_k \partial_i u^n$$

- *Total disclination charge:*

$$-\frac{2\pi}{n} s = \int d^2x \, (\rho_s - \epsilon_{i\ell} \partial^i \rho_b^\ell)$$

FRACTON TENSOR GAUGE THEORY

- ***Governed by the Gauss's law***

$$\partial_i \partial_j E^{ij} = \rho$$

- ***Governed by the gauge transformation***

$$A_{ij} \rightarrow A_{ij} + \partial_i \partial_j \alpha$$

- ***Governed by the conservation of charge***

$$q = \int_V d^2x\, \rho = \int_V d^2x\, \partial_i \partial_j E^{ij} = \int_{\partial V} dn_i\, \partial_j E^{ij}$$

- ***Additionally, there is a conservation of dipole moment***

$$P^i = \int_V d^2x\, (\rho x^i) = \int_V d^2x\, x^i \partial_j \partial_k E^{jk}$$
$$= \int_{\partial V} dn_j\, (x^i \partial_k E^{jk} - E^{ij}).$$

FRACTON TENSOR GAUGE THEORY (continued)

- *This extra conservation law has dramatic consequences for the particles of the theory: an isolated charge is strictly locked in place. Only neutral bound states, such as dipoles, can move around the system.*

- *The dipolar conservation law implies that the dipoles of the theory are topologically stable excitations, despite being charge-neutral.*

- *Write down the most general gauge-invariant Hamiltonian for the charge-free sector*

$$H = \int d^2x \left(\frac{1}{2}\tilde{C}^{ijk\ell}E_{ij}E_{k\ell} + \frac{1}{2}B^iB_i \right)$$

- *Performing a canonical transformation*

$$S = \int d^2x dt \left(\frac{1}{2}\tilde{C}^{-1}_{ijk\ell}E^{ij}_{\sigma}E^{k\ell}_{\sigma} - \frac{1}{2}B^iB_i \right)$$

$$\tilde{C}^{-1}_{ijk\ell}\tilde{C}^{k\ell mn} = \delta_{ij}\delta^{mn}$$

$$E^{ij}_{\sigma} = -\partial_t A^{ij} - \partial^i\partial^j\phi,$$

FRACTON TENSOR GAUGE THEORY (continued)

- *This new variable is actually related by a tensor*

$$E_{ij} = -\frac{\partial \mathcal{L}}{\partial \dot{A}^{ij}} = \tilde{C}^{-1}_{ijk\ell} E^{k\ell}_{\sigma}.$$

- *These are invariant under time-dependent gauge transformations*

$$A_{ij} \to A_{ij} + \partial_i \partial_j \alpha, \qquad \phi \to \phi + \partial_t \alpha$$

- *Generalized Faraday's equation of the theory*

$$\partial_t B^i + \epsilon_{jk} \partial^j E^{ki}_{\sigma} = 0,$$

- *In the presence of fracton charges coupled to the gauge field*

$$S = \int d^2 x dt \left(\frac{1}{2} \tilde{C}^{-1}_{ijk\ell} E^{ij}_{\sigma} E^{k\ell}_{\sigma} - \frac{1}{2} B^i B_i - \rho \phi - J^{ij} A_{ij} \right)$$

DERIVATION OF DUALITY

- *The two-dimensional elasticity theory action:*

$$S = \int d^2x dt \, \tfrac{1}{2} \left[(\partial_t u^i)^2 - C^{ijk\ell} u_{ij} u_{k\ell} \right]$$

- *How the strain responds to the presence of disclinations:*

$$\epsilon^{ik} \epsilon^{j\ell} \partial_i \partial_j u_{k\ell} = \rho.$$

- *A dislocation can be regarded as a bound state of two disclinations*
- *Separate the displacement field into its singular and smooth phonon pieces* $u_{ij} = u_{ij}^{(s)} + \tfrac{1}{2}(\partial_i \tilde{u}_j + \partial_j \tilde{u}_i)$
- *Fields obey*

$$\epsilon^{ik} \epsilon^{j\ell} \partial_i \partial_j \tilde{u}_{k\ell} = 0$$

$$\epsilon^{ik} \epsilon^{j\ell} \partial_i \partial_j u_{k\ell}^{(s)} = \rho_s$$

HUBBARD–STRATANOVICH TRANSFORMATION

- *Introduce two Hubbard-Stratonovich fields, the lattice momentum* π_i *and the stress tensor* σ_{ij}

- *Rewrite the action as:*

$$S = \int d^2x dt \left[\frac{1}{2} C^{-1}_{ijk\ell} \sigma^{ij} \sigma^{k\ell} - \frac{1}{2} \pi^i \pi_i - \sigma^{ij} (\partial_i \tilde{u}_j + u^{(s)}_{ij}) + \pi^i \partial_t (\tilde{u}_i + u^{(s)}_i) \right]$$

- *The smooth displacement field can be integrated out to enforce the constraint:*

$$\partial_t \pi^i - \partial_j \sigma^{ij} = 0$$

ROTATED FIELDS

- *Introduce rotated fields*

$$B^i = \epsilon^{ij}\pi_j$$

$$E_\sigma^{ij} = \epsilon^{ik}\epsilon^{j\ell}\sigma_{k\ell}$$

- *The constraint equation becomes a generalized Faraday's law*

$$\partial_t B^i + \epsilon_{jk}\partial^j E_\sigma^{ki} = 0.$$

- *This is precisely the form in the Tensor Gauge Theory with*

$$B^i = \epsilon_{jk}\partial^j A^{ki}\,, \qquad E_\sigma^{ij} = -\partial_t A^{ij} - \partial_i\partial_j\phi\ .$$

FINAL ACTION

- ***Additional gauge transformation introduced***

$$A_{ij} \to A_{ij} + \partial_i \partial_j \alpha \,, \qquad \phi \to \phi + \partial_t \alpha,$$

- ***Utilizing these potentials, the action can be written as:***

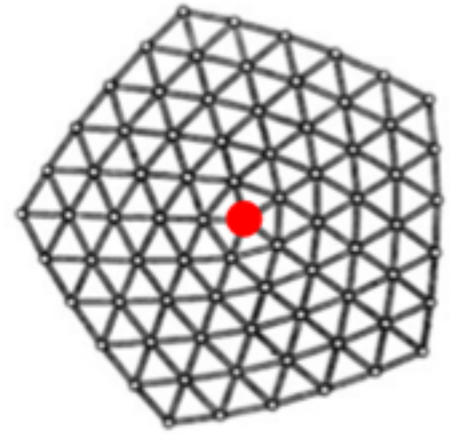
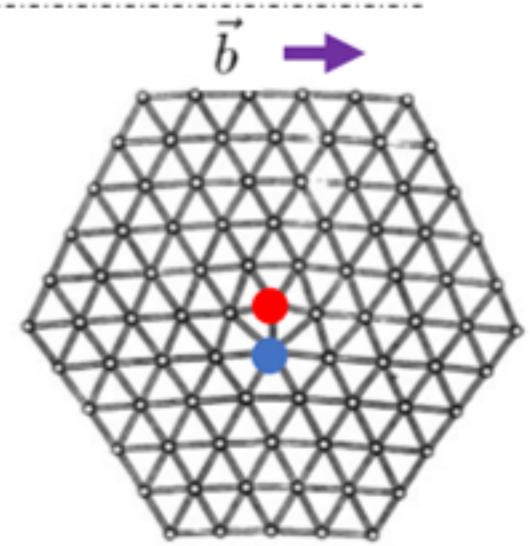
$$S = \int d^2x dt \Bigg(\tfrac{1}{2} \tilde{C}^{-1}_{ijk\ell} E^{ij}_{\sigma} E^{k\ell}_{\sigma} - \tfrac{1}{2} B^i B_i$$
$$+ \epsilon^{ik} \epsilon^{j\ell} \partial_t (A_{k\ell} + \partial_k \partial_\ell \phi) u^{(s)}_{ij} - \epsilon^{ij} \epsilon_{k\ell} \partial^k A^{\ell j} \partial_t u^{(s)}_i \Bigg)$$

with $\tilde{C}^{ijk\ell} = \epsilon^{ia} \epsilon^{jb} \epsilon^{kc} \epsilon^{\ell d} C_{abcd}$

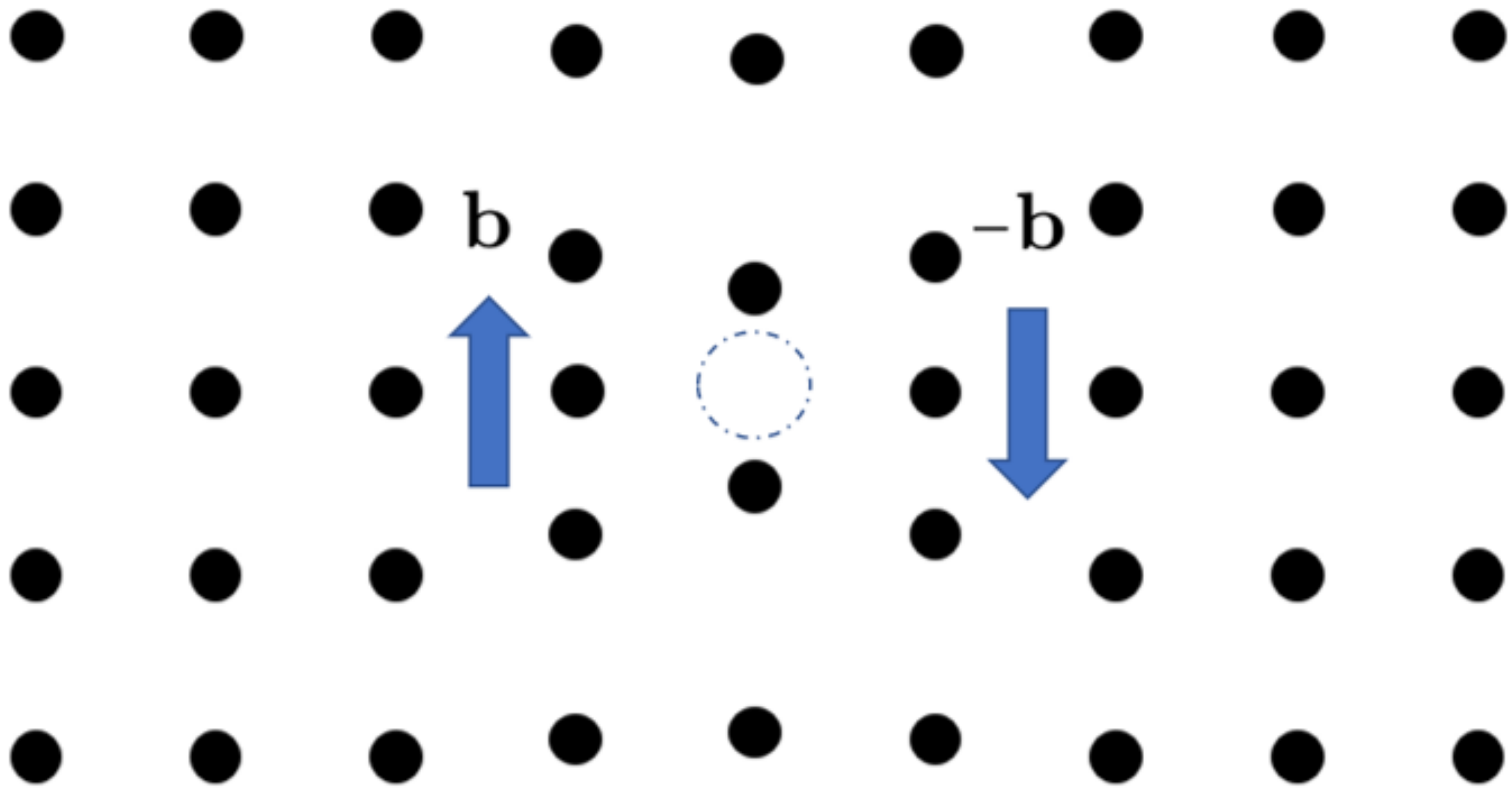
- ***Integrating the last two terms by parts, we finally obtain***

$$\boxed{S = \int d^2x dt \Bigg(\tfrac{1}{2} \tilde{C}^{-1}_{ijk\ell} E^{ij}_{\sigma} E^{k\ell}_{\sigma} - \tfrac{1}{2} B^i B_i + \rho \phi - J^{ij} A_{ij} \Bigg)}$$

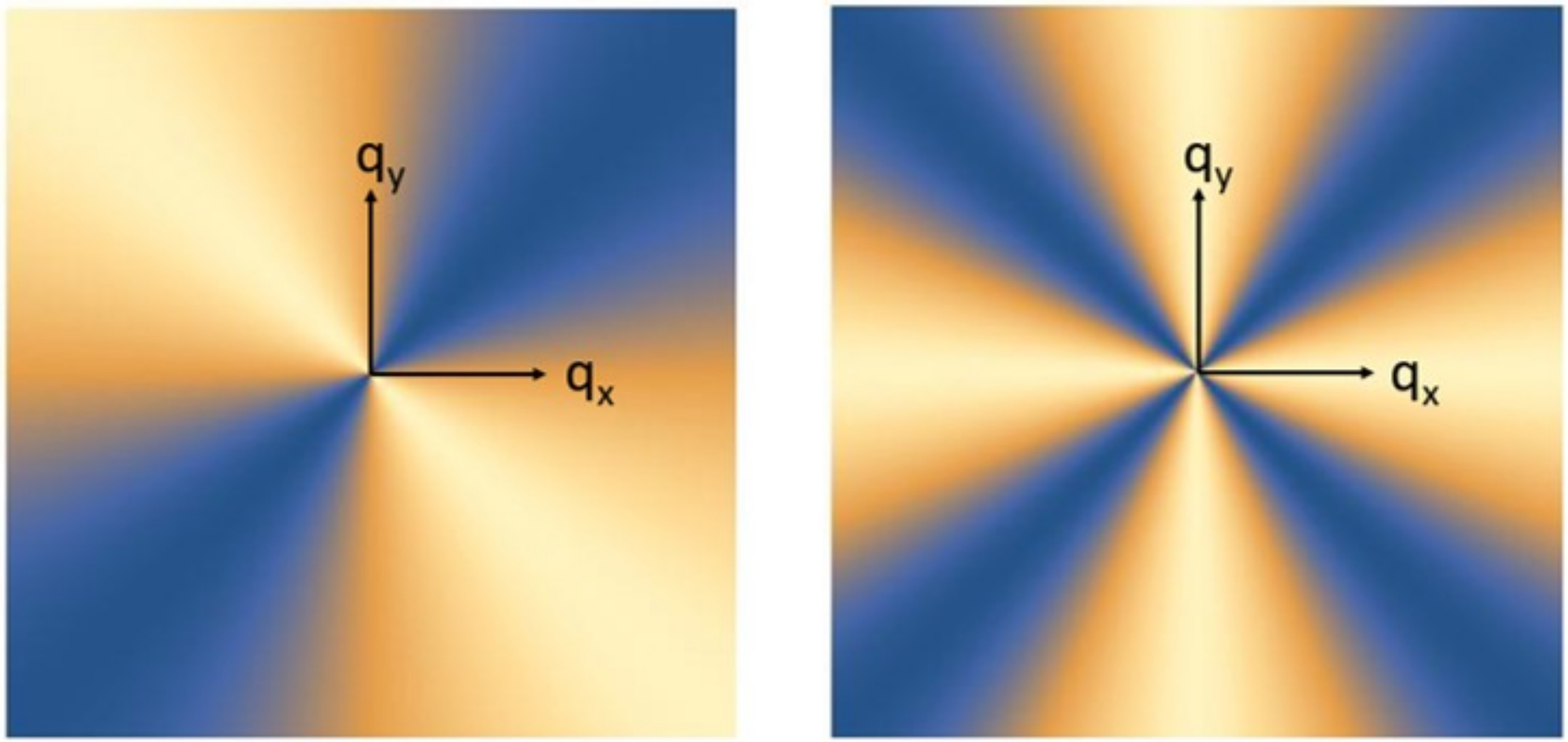
DICTIONARY

Fracton $\partial_i \partial_j E^{ij} = \rho$	Disclination $\epsilon^{ik}\epsilon^{j\ell}\partial_i\partial_j u_{k\ell} = s$ 
Dipole	 Dislocation
Gauge Modes	Phonons
Electric Field E_{ij}	Strain Tensor u_{ij}
Magnetic Field B_i	Lattice Momentum π_i

MOTION OF DEFECTS

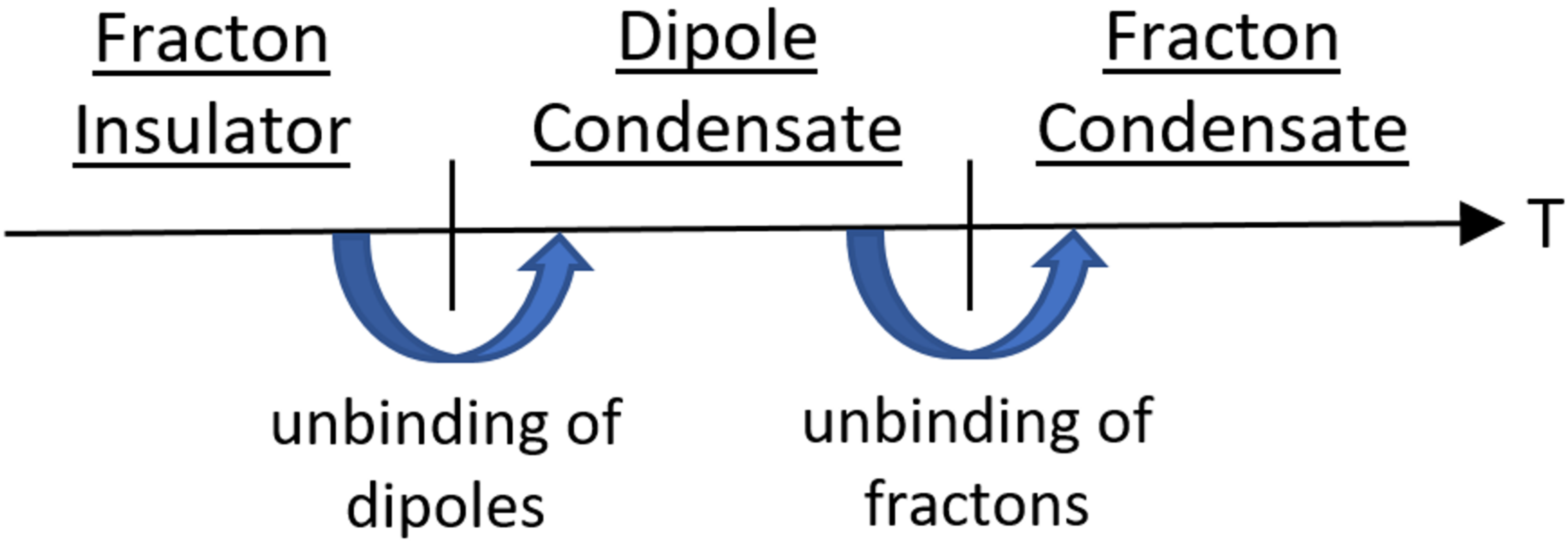


PINCH-POINT SINGULARITIES



At left is a schematic plot of $\langle E^x(q)E^y(-q)\rangle$ *in the* $q_x - q_y$ *plane, displaying the characteristic two-fold pinch-point singularity of conventional gauge theories. At right is an analogous plot of* $\langle E^{xx}(q)E^{yy}(-q)\rangle$ *displaying the four-fold pinch-point singularity of a rank-2 tensor gauge theory.*

PHASE DIAGRAM



Thank You.